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The onset of bio-thermal convection in a suspension of gyrotactic microorganisms in a fluid layer: Oscillatory convection

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Abstract

A linear stability analysis is applied to investigate the onset of bioconvection in a horizontal layer of fluid containing a suspension of motile microorganisms with heating or cooling from below. With cooling from below the stabilizing effect of the thermal stratification is opposed to the destabilizing effect resulting from the congregation of the microorganisms, and oscillatory convection is possible in certain circumstances. The stability criterion is found in terms of a thermal Rayleigh number, a bioconvection Rayleigh number, a bioconvection Péclet number, a gyrotactic number, and a measure of the cell eccentricity, together with (in the case of oscillatory convection) a Prandtl number and a Lewis number.

Keywords: Oscillatory bioconvection; Gyrotactic microorganisms; Heated layer

1. Introduction

The term bioconvection refers to macroscopic convection induced in water by the collective motion of a large number of self-propelled motile microorganisms that leads to an unstable density stratification [1–7]. With possible application to the dynamics of thermophilic organisms in mind, Kuznetsov [8] presented a linear stability analysis of a suspension of gyrotactic microorganisms in an isothermal fluid layer of finite depth heated from below. In this set of circumstances the thermal stratification and the stratification due to the swimming of the organisms are both destabilizing, and when instability results it is of the non-oscillatory kind. The purpose of the present paper is to complement the above paper by a study of the case where the layer is cooled from below, so that the thermal stratification is stabilizing and hence opposes the bioconvection effect. In such circumstances overstability (the onset of instability of an oscillatory kind) is possible.

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2. Problem formulation and analysis

2.1. Governing equations

The present analysis is a modified version of that presented in Hill et al. [9]. It is assumed that heating from below is sufficiently weak, so it does not kill microorganisms and does not affect their gyrotactic behavior. Inertia terms in the Navier–Stokes equations are neglected for a linear stability analysis at the onset of bioconvection [1,2]. The model presented here is based on a continuum model of a suspension of gyrotactic microorganisms developed in Pedley et al. [1]. This model is supplemented by an energy equation and a buoyancy term in the momentum equation that results from the temperature variation across the layer. The geometry shown in Fig. 1 is considered. The Boussinesq approximation is utilized. Under these assumptions, the governing equations can be presented as:

$$\rho_w \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{g}\big[n\theta\Delta\rho - \rho_w \beta (T - T_0)\big] \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

$$c_p \rho_w \left(\frac{\partial T}{\partial t} + \mathbf{v} \cdot \nabla T\right) = k \nabla^2 T \tag{3}$$

Nomenclature

a	semi-major axis of the spheroidal cell
b	semi-minor axis of the spheroidal cell
B	gyrotactic orientation parameter, $\alpha_{\perp}\mu/(2h\rho_0 g)$
c_p	specific heat of water
D	diffusivity of microorganisms
$\mathbf{g}$	gravity vector
G	gyrotaxis number, BD/H^2
H	layer depth
$\mathbf{j}$	flux of microorganisms
k	thermal conductivity of water
Le	Lewis number, $k/(Dc_p\rho_w)$
n	number density of motile microorganisms
N	amplitude of number density of microorganisms in the normal mode expansion given by Eq. (29)
p	excess pressure (above hydrostatic)
$\hat{\mathbf{p}}$	unit vector indicating the direction of swimming of microorganisms
Pr	Prandtl number, $\mu c_p/k$
Q	bioconvection Péclet number, $W_c H/D$
R	traditional Rayleigh number, $(g\beta\Delta T H^3\rho_w^2 c_p)/(\mu k)$
S	bioconvection Rayleigh number, $(Q\Delta\rho g\nu\theta H^3)/(\mu D)$
t	time
T	temperature
T_0	temperature at the upper surface
u, v, w	velocity components
W	vertical velocity amplitude in the normal mode expansion given by Eq. (29)
$\mathbf{v}$	fluid convection velocity vector
W_c	average swimming velocity of microorganisms
x, y, z	Cartesian coordinates (z is the vertical coordinate)

Greek symbols

α_0	measure of the cell eccentricity, $(a^2-b^2)/(a^2+b^2)$
β	volume expansion coefficient of water at constant pressure
$\Delta\rho$	density difference, $\rho_{\text{cell}}-\rho_w$
θ	average volume of a microorganism
Θ	temperature amplitude in the normal mode expansion given by Eq. (29)
μ	dynamic viscosity of the suspension
ρ_w	density of water
σ	dimensionless amplification factor
Ω	measure of density of microorganisms, $\Delta\rho/\rho_w$

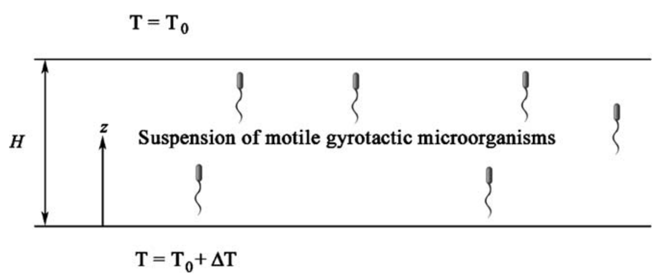

Fig. 1. Schematic sketch for the problem.

$$\frac{\partial n}{\partial t} = -\,\mathrm{div}(\mathbf{j}) \tag{4}$$

where

$$\mathbf{j} = n\mathbf{v} + nW_c\hat{\mathbf{p}} - D\nabla n \tag{5}$$

is the flux of microorganisms due to macroscopic convection of the fluid, self-propelled swimming of microorganisms, and diffusion of microorganisms (as in [1], it is assumed that all random motions of microorganisms can be approximated by a diffusive process), respectively; c_p is the specific heat of water; D is the diffusivity of microorganisms; $\mathbf{g}$ is the gravity vector; k is the thermal conductivity of water; n the number density of motile microorganisms; p is the excess pressure (above hydrostatic); $\hat{\mathbf{p}}$ is the unit vector indicating the direction of swimming of microorganisms; t is the time; T is the temperature; u, v, and w are the x, y, and z-velocity components, respectively; $\mathbf{v}$ is the fluid convection velocity vector, (u, v, w); $W_c\hat{\mathbf{p}}$ is the vector of average swimming velocity relative to the fluid (W_c is assumed to be constant); x, y, and z are the Cartesian coordinates (z is the vertical coordinate); β is the volume expansion coefficient of water at constant pressure; $\Delta\rho = \rho_{\text{cell}} - \rho_w$ is the density difference; θ is the average volume of a microorganism; μ is the dynamic viscosity of the suspension, assumed to be approximately the same as that of water; and ρ_w is the density of water.

2.2. Boundary conditions

A schematic diagram illustrating the problem is displayed in Fig. 1. A horizontal fluid layer of depth H is considered. Cartesian axes with the z-axis vertical are utilized, so that the layer is confined between $z = 0$ and $z = H$. It is assumed that the layer is unbounded in the x and y directions.

At the bottom of the layer (assumed to be rigid) the following conditions are satisfied:

$$\text{At } z = 0: \quad u = v = w = 0,\ T = T_0 + \Delta T,\ \mathbf{j}\cdot\hat{\mathbf{k}} = 0 \tag{6}$$

where $\hat{\mathbf{k}}$ is the vertically-upward unit vector.

The upper surface of the layer is assumed rigid as well because, according to Hill et al. [9], even if it is open to the air, microorganisms tend to collect at the surface forming what appears to be a packed layer, and it is unlikely that the upper boundary is ever fully stress free. Under this assumption, the boundary conditions at the upper surface of the layer are:

$$\text{At } z = H: \quad u = v = w = 0,\ T = T_0,\ \mathbf{j}\cdot\hat{\mathbf{k}} = 0 \tag{7}$$

2.3. Basic state

The equation of continuity admits a steady-state solution where the fluid is motionless and n_b, the number density of the microorganisms in the basic state, p_b, the pressure in the basic state, and T_b, the temperature in the basic state, are functions of z only.

In this case Eqs. (4) and (5) reduce to

$$n_b W_c = D\frac{\partial n_b}{\partial z} \tag{8}$$

The solution of this equation is

$$n_b(z) = \nu \exp\left(\frac{W_c z}{D}\right) \tag{9}$$

The integration constant ν, which represents the value of the basic number density at the bottom of the layer, is related to the average concentration $\bar{n}$ by

$$\bar{n} = \frac{1}{H}\int_0^H n_b(z)\,\mathrm{d}z = \frac{\nu}{H}\int_0^H \exp\left(\frac{W_c z}{D}\right)\mathrm{d}z \tag{10}$$

and so is given by

$$\nu = \frac{\bar{n}Q}{\exp(Q) - 1} \tag{11}$$

where the bioconvection Péclet number Q is defined by

$$Q = \frac{W_c H}{D} \tag{12}$$

From Eqs. (3), (6), (7), the temperature distribution in the basic state is:

$$T_b = T_0 + \Delta T\left(1 - \frac{z}{H}\right) \tag{13}$$

Finally, from Eq. (1) the pressure distribution in the basic state is found from integrating the following equation:

$$\frac{\partial p}{\partial z} = -\nu\theta\Delta\rho g \exp\left(\frac{W_c z}{D}\right) + \rho_w g\beta\Delta T\left(1 - \frac{z}{H}\right) \tag{14}$$

Assuming that $p = p_0$ at $z = H$, the pressure distribution in the basic state is

$$p_b = p_0 + \nu\theta\Delta\rho g\frac{D}{W_c}\left[\exp(Q) - \exp\left(\frac{W_c z}{D}\right)\right] - \rho_w g\beta\Delta T\left(H - z - \frac{1}{2H}\left(H^2 - z^2\right)\right) \tag{15}$$

2.4. Linear stability analysis

The perturbations are introduced as follows:

$$[n, \mathbf{v}, p, T, \hat{\mathbf{p}}] = \left[n_b(z), 0, p_b(z), T_b(z), \hat{\mathbf{k}}\right] + \varepsilon\left[n^*(t,x,y,z), \mathbf{v}^*(t,x,y,z), p^*(t,x,y,z), T^*(t,x,y,z), \hat{\mathbf{p}}^*(t,x,y,z)\right] \tag{16}$$

where a star denotes a perturbation quantity, and ε is the small perturbation amplitude. Substituting Eq. (16) into Eqs. (1)–(5) and linearizing results in the following equations for perturbations:

$$\rho_w\frac{\partial \mathbf{U}^*}{\partial t} = -\nabla p^* + \mu\nabla^2\mathbf{v}^* + \mathbf{g}\left[n^*\theta\Delta\rho - \rho_w\beta T^*\right] \tag{17}$$

$$\nabla\cdot\mathbf{v}^* = 0 \tag{18}$$

$$c_p\rho_w\left(\frac{\partial T^*}{\partial t} - w^*\frac{\Delta T}{H}\right) = k\nabla^2 T^* \tag{19}$$

$$\frac{\partial n^*}{\partial t} = -\operatorname{div}\left[n_0(\mathbf{v}^* + W_c\hat{\mathbf{p}}^*) + n^* W_c\hat{\mathbf{k}} - D\nabla n^*\right] \tag{20}$$

The elimination of u^*, v^*, and p^* from Eqs. (17)–(18) results in:

$$\rho_w\frac{\partial}{\partial t}\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} + \frac{\partial^2 w^*}{\partial z^2}\right) = -\theta\Delta\rho g\left(\frac{\partial^2 n^*}{\partial x^2} + \frac{\partial^2 n^*}{\partial y^2}\right) + \rho_w g\beta\left(\frac{\partial^2 T^*}{\partial x^2} + \frac{\partial^2 T^*}{\partial y^2}\right) + \mu\left(\frac{\partial^4 w^*}{\partial x^4} + \frac{\partial^4 w^*}{\partial y^4} + \frac{\partial^4 w^*}{\partial z^4} + 2\frac{\partial^4 w^*}{\partial x^2\partial y^2} + 2\frac{\partial^4 w^*}{\partial x^2\partial z^2} + 2\frac{\partial^4 w^*}{\partial y^2\partial z^2}\right) \tag{21}$$

Since it is assumed that temperature variation within the fluid layer does not influence gyrotactic behavior of microorganisms, according to Pedley et al. [2], for gyrotactic microorganisms,

$$\hat{\mathbf{p}}^* = B(\eta, -\xi, 0) \tag{22}$$

where

$$\xi = (1 - \alpha_0)\frac{\partial w^*}{\partial y} - (1 + \alpha_0)\frac{\partial v^*}{\partial z} \tag{23}$$

$$\eta = -(1 - \alpha_0)\frac{\partial w^*}{\partial x} + (1 + \alpha_0)\frac{\partial u^*}{\partial z} \tag{24}$$

$$\alpha_0 = \frac{a^2 - b^2}{a^2 + b^2} \tag{25}$$

$$B = \frac{\alpha_\perp\mu}{2h\rho_0 g} \tag{26}$$

where a and b are the semi-major and semi-minor axes of the spheroidal cell, so α_0 is a measure of the cell eccentricity; B is the "gyrotactic orientation parameter" which was introduced by Pedley and Kessler [10] and which has dimensions of time; $\alpha_\perp$ is a dimensionless constant relating viscous torque to the relative angular velocity of the cell; and h is the displacement of the center of mass of the cell from the center of buoyancy.

Thus Eq. (20) can be rewritten as follows:

$$\frac{\partial n^*}{\partial t} = -w^*\frac{\partial n_b}{\partial z} - W_c\frac{\partial n^*}{\partial z} - W_c B n_b\left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial y}\right) + D\nabla^2 n^* \tag{27}$$

Accounting for Eqs. (23) and (24), Eq. (27) can be recast as:

$$\frac{\partial n^*}{\partial t} = -w^*\frac{\partial n_b}{\partial z} - W_c\frac{\partial n^*}{\partial z} + W_c B n_b\left((1 - \alpha_0)\left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2}\right) + (1 + \alpha_0)\frac{\partial^2 w^*}{\partial z^2}\right) + D\nabla^2 n^* \tag{28}$$

A normal mode expansion is introduced in the following form:

$$[w^*, n^*, T^*] = \big[W(z), N(z), \Theta(z)\big] f(x,y)\exp(\sigma t) \tag{29}$$

The function $f(x,y)$ satisfies the following equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -m^2 f \tag{30}$$

where m the is the horizontal wavenumber (used as a separation constant).

Substituting Eq. (29) into Eqs. (19), (21), (28), and accounting for Eq. (30), the following equations for the amplitudes W, Θ, and N, are obtained:

$$-\theta\Delta\rho g m^2 N - \big(\mu m^4 + \rho_w\sigma m^2\big)W + \rho_w g\beta m^2\Theta + \big(2\mu m^2 + \rho_w\sigma\big)W'' - \mu W^{IV} = 0 \tag{31}$$

$$-c_p\Delta T\rho_w W + H\big[\big(km^2 + c_p\rho_w\sigma\big)\Theta - k\Theta''\big] = 0 \tag{32}$$

$$D\big(Dm^2+\sigma\big)N + D(W_c N' - DN'') - \exp\left(\frac{W_c z}{D}\right)W_c\nu\big[-\big(1+BDm^2(1-\alpha_0)\big)W + BD(1+\alpha_0)W''\big] = 0 \tag{33}$$

Introducing the following dimensionless variables,

$$\bar{z} = \frac{z}{H}, \qquad a = mH, \qquad \bar{W} = \frac{\nu\theta W_c H^2}{D^2}W,$$
$$\bar{N} = N\theta, \qquad \bar{\Theta} = \beta\Theta, \qquad s = \frac{\rho_w H^2}{\mu}\sigma \tag{34}$$

Eqs. (31)–(33) can be recast as:

$$-Sa^2\bar{N} - \big(a^4 + a^2 s\big)\bar{W} + \frac{Sa^2}{\Omega}\bar{\Theta} + \big(2a^2+s\big)\bar{W}'' - \bar{W}^{IV} = 0 \tag{35}$$

$$-\frac{\Omega R}{S}\bar{W} + \big(a^2 + Pr\,\sigma\big)\bar{\Theta} - \bar{\Theta}'' = 0 \tag{36}$$

$$\big(a^2 + Pr\,Le\,s\big)\bar{N} + (Q\bar{N}' - \bar{N}'') - \exp(Q\bar{z})\big[-\big(1+G(1-\alpha_0)a^2\big)\bar{W} + G(1+\alpha_0)\bar{W}''\big] = 0 \tag{37}$$

where $R = \frac{g\beta\Delta T H^3\rho_w^2 c_p}{\mu k}$ is the traditional Rayleigh number associated with natural convection, $S = \frac{Q\Delta\rho g\nu\theta H^3}{\mu D}$ is the bioconvection Rayleigh number, $\Omega = \frac{\Delta\rho}{\rho_w}$ is the measure of density of microorganisms, $G = \frac{BD}{H^2}$ is the gyrotaxis number, $Pr = \frac{\mu c_p}{k}$ is the Prandtl number, and $Le = \frac{k}{Dc_p\rho_w}$ is the Lewis number. (In [8] R is denoted by Ra and S is denoted by $\tilde{R}b$, that is by QRb.)

Since both lower and upper boundaries of the layer are assumed rigid, Eqs. (35)–(37) must be solved subject to the following boundary conditions:

$$\text{At } z=0\text{:}\quad W=0,\ \frac{d\bar{W}}{d\bar{z}}=0,\ \bar{\Theta}=0,\ Q\bar{N}=\frac{d\bar{N}}{d\bar{z}} \tag{38}$$

$$\text{At } z=H\text{:}\quad W=0,\ \frac{d\bar{W}}{d\bar{z}}=0,\ \bar{\Theta}=0,\ Q\bar{N}=\frac{d\bar{N}}{d\bar{z}} \tag{39}$$

At the onset of convection $s = i\omega$, where the frequency ω is real.

For the solution of this system, a simple Galerkin method is employed. Suitable trial functions (satisfying the boundary conditions) are:

$$\bar{W}_1 = \bar{z}^2 - 2\bar{z}^3 + \bar{z}^4 \tag{40}$$

$$\bar{\Theta}_1 = \bar{z} - \bar{z}^2 \tag{41}$$

$$\bar{N}_1 = 2 - Q(1-2\bar{z}) - Q^2\big(\bar{z}-\bar{z}^2\big) \tag{42}$$

The utilization of a standard Galerkin procedure [11] results in the following equation for the stability boundary.

$$\begin{vmatrix} \frac{4}{5}+\frac{4a^2}{105}+\frac{a^4}{630} + i\omega\left(\frac{2}{105}+\frac{a^2}{630}\right) & -\frac{a^2 S}{140\Omega} & a^2 S\left(\frac{1}{15}-\frac{Q^2}{140}\right) \\ -\frac{R\Omega}{140S} & \frac{1}{3}+\frac{a^2}{30}+\frac{i\,Pr\,\omega}{30} & 0 \\ -8F(Q,G,\alpha_0) & 0 & \frac{Q^4}{3} + (a^2 + i\,Pr\,Le\,\omega)\times\left(4-\frac{Q^2}{3}+\frac{Q^4}{30}\right) \end{vmatrix} = 0 \tag{43}$$

where

$$F(Q,G,\alpha_0) = e^{Q/2}Q^{-5}(f_1+f_2) \tag{44}$$

where in turn

$$f_1 = Q\big[66 + Q^2 - G\big\{a^2\big(66+Q^2\big)(-1+\alpha_0) + 24Q^2(1+\alpha_0)\big\}\big]\cosh(Q/2) \tag{45}$$

$$f_2 = \big[-132 - 13Q^2 + G\big\{a^2\big(132+13Q^2\big)(-1+\alpha_0) + 4Q^2\big(12+Q^2\big)(1+\alpha_0)\big\}\big]\sinh(Q/2) \tag{46}$$

It is noteworthy that the density ratio Ω cancels out when the determinant in Eq. (43) is expanded.

Eq. (43) can be written in the form

$$(\alpha_1 + i\beta_1\omega)R + (\alpha_2 + i\beta_2\omega)S = (\alpha_3 + i\beta_3\omega)(\alpha_4 + i\beta_4\omega)(\alpha_5 + i\beta_5\omega) \tag{47}$$

where

$$\alpha_1 = 27a^2\big[10Q^4 + a^2\big(120 - 10Q^2 + Q^4\big)\big]$$
$$\beta_1 = 27\,Pr\,Le\,a^2\big(120 - 10Q^2 + Q^4\big)$$
$$\alpha_2 = 10080a^2\big(10+a^2\big)\big(3Q^2-28\big)F(Q,G,\alpha_0)$$
$$\beta_2 = 10080\,Pr\,a^2\big(3Q^2-28\big)F(Q,G,\alpha_0)$$
$$\alpha_3 = 28\big(504 + 24a^2 + a^4\big)$$
$$\beta_3 = 28\big(12+a^2\big)$$
$$\alpha_4 = 10 + a^2$$
$$\beta_4 = Pr$$
$$\alpha_5 = 10Q^4 + a^2\big(120 - 10Q^2 + Q^4\big)$$
$$\beta_5 = Pr\,Le\big(120 - 10Q^2 + Q^4\big)$$

For monotonic convection ($\omega = 0$), Eq. (47) reduces to

$$\alpha_1 R + \alpha_2 S = \alpha_3\alpha_4\alpha_5 \tag{48}$$

For oscillatory convection ($\omega \neq 0$), the real and imaginary parts of Eq. (47) yield

$$\omega^2 = \frac{\alpha_3\alpha_4\alpha_5 - \alpha_1 R - \alpha_2 S}{\alpha_3\beta_4\beta_5 + \alpha_4\beta_5\beta_3 + \alpha_5\beta_3\beta_4} = \frac{\alpha_4\alpha_5\beta_3 + \alpha_5\alpha_3\beta_4 + \alpha_3\alpha_4\beta_5 - \beta_1 R - \beta_2 S}{\beta_3\beta_4\beta_5} \tag{49}$$

The second equality in Eq. (49) gives the oscillatory instability boundary as

$$(\alpha_3\beta_1\beta_4\beta_5 + \alpha_4\beta_1\beta_3\beta_5 + \alpha_5\beta_1\beta_3\beta_4 - \alpha_1\beta_3\beta_4\beta_5)R + (\alpha_3\beta_2\beta_4\beta_5 + \alpha_4\beta_2\beta_3\beta_5 + \alpha_5\beta_2\beta_3\beta_4 - \alpha_2\beta_3\beta_4\beta_5)S = (\alpha_4\alpha_5\beta_3 + \alpha_3\alpha_5\beta_4 + \alpha_3\alpha_4\beta_5)(\alpha_3\beta_4\beta_5 + \alpha_4\beta_3\beta_5 + \alpha_5\beta_3\beta_4) - \alpha_3\alpha_4\alpha_5\beta_3\beta_4\beta_5 \tag{50}$$

It is clear that in order to have a real nonzero value of ω, R and S must be of opposite sign. Since S has to be positive on physical grounds, that means that R has to be negative for oscillatory convection to be possible.

Some simplification can be made by introducing the notation

$$\rho_{i}=\alpha_i/\beta_i \quad (i = 1, 2, 3, 4, 5)$$
$$R_0 = \alpha_3\alpha_4\alpha_5/\alpha_1, \qquad S_0 = \alpha_3\alpha_4\alpha_5/\alpha_2 \tag{51}$$

and using the identities $\rho_4 = \rho_2$ and $\rho_5 = \rho_1$.

The monotonic stability boundary is then

$$\frac{R}{R_0} + \frac{S}{S_0} = 1 \tag{52}$$

the oscillatory stability boundary is

$$\frac{R}{R_0}\left[\rho_1\left(\frac{1}{\rho_2} + \frac{1}{\rho_3}\right)\right] + \frac{S}{S_0}\left[\rho_2\left(\frac{1}{\rho_1} + \frac{1}{\rho_3}\right)\right] = (\rho_1 + \rho_2 + \rho_3)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2} + \frac{1}{\rho_3}\right) - 1 \tag{53}$$

and the frequency ω of oscillations is given by

$$\omega^2 = \frac{1 - R/R_0 - S/S_0}{\rho_2\rho_3 + \rho_3\rho_1 + \rho_1\rho_2} = \frac{\rho_1 + \rho_2 + \rho_3 - \rho_1 R/R_0 - \rho_2 S/S_0}{\rho_1\rho_2\rho_3} \tag{54}$$

In the limiting case where Q tends to zero one has

$$\rho_1 = \rho_5 = \frac{a^2}{Pr\,Le}$$
$$\rho_2 = \rho_4 = \frac{10 + a^2}{Pr}$$
$$\rho_3 = \frac{504 + 24a^2 + a^4}{12 + a^2} \tag{55}$$

$$R_0 = \frac{28(504 + 24a^2 + a^4)(10 + a^2)}{27a^2} \tag{56}$$

$$S_0 = \frac{10(504 + 24a^2 + a^4)}{7[1 + Ga^2(1 - \alpha_0)]} \tag{57}$$

As the wavenumber a varies, the minimum value of R_0 as given by Eq. (56) is 1750, a value about 2.5% higher than the well-known precise value 1707.762 given in [12]. This is an indication of the accuracy expected from the one-term Galerkin approximation. The minimum value is attained when $a = 3.12$.

Similarly, for the case when G is zero, the minimum value of S_0 as given by Eq. (57) is 720, a known exact result for a critical Rayleigh number (see, for example, the appendix to [13]). The minimum value is attained when $a = 0$.

3. Results and discussion

There are a large number of dimensionless parameters involved. Besides the Rayleigh numbers R and S, the wavenumber a and the frequency ω, we have the parameters Q, G, α_0, Pr, and Le. We present the results of calculations for the typical values $\alpha_0 = 0.2$, $Pr = 7$ and $Le = 1/3$. A typical situation is illustrated in Fig. 2, for the case $G = 1$, $Q = 1$. Oscillatory

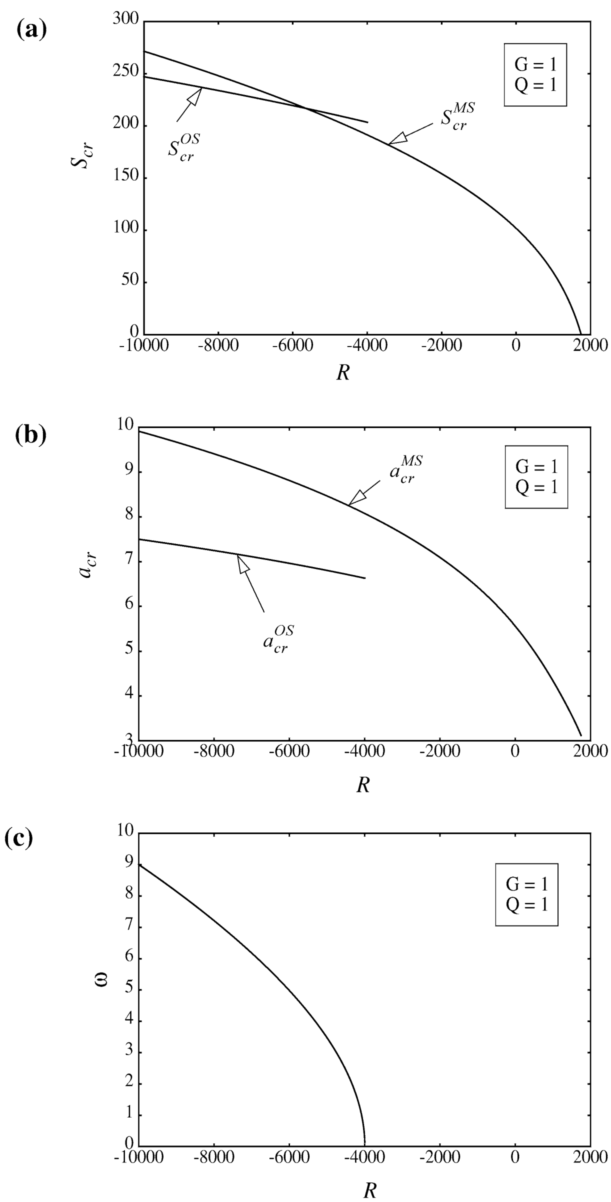

Fig. 2. Plots of (a) critical bioconvection Rayleigh number S versus thermal Rayleigh number R, for the monotonic stability (MS) and oscillatory stability (OS) modes of convection, (b) critical wavenumber a versus R, (c) oscillation frequency ω versus R, for the case $G = 1$, $Q = 1$, where G is a gyrotactic number and Q is a Péclet number.

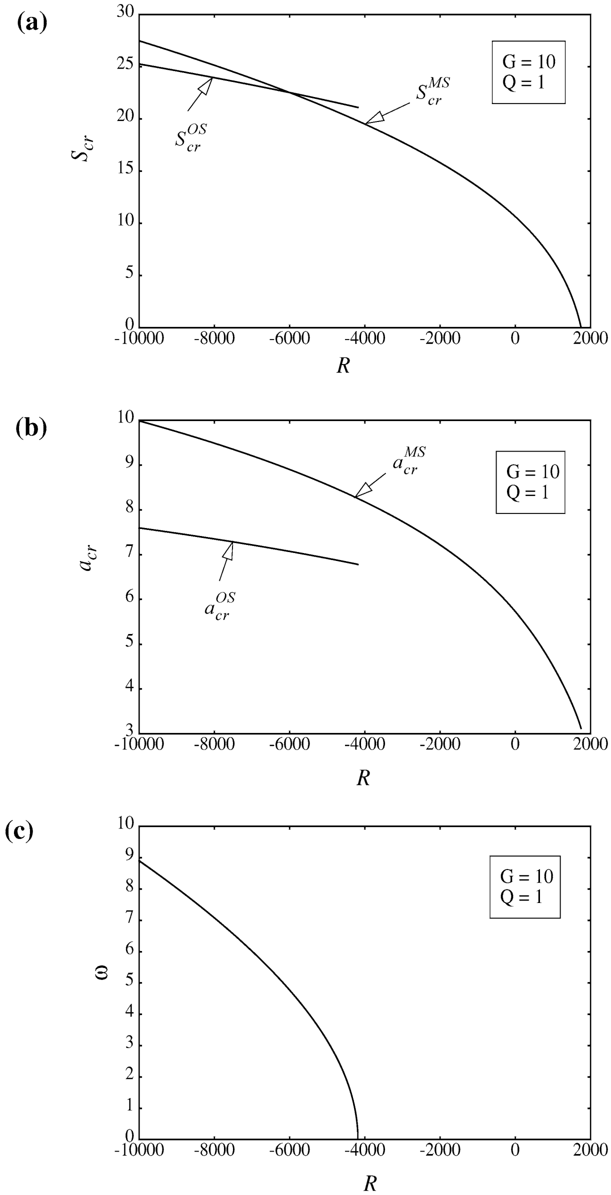

Fig. 3. As for Fig. 2, but with $G = 10$ instead of $G = 1$.

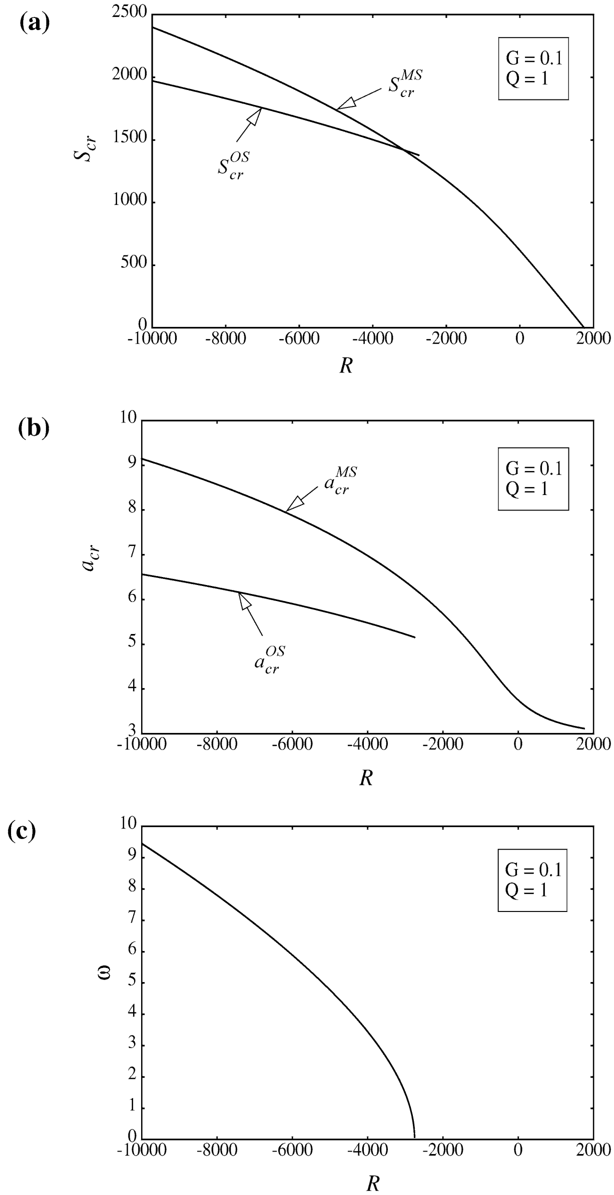

Fig. 4. As for Fig. 2, but with $G = 0.1$ instead of $G = 1$.

convection is possible only for values of R that are sufficiently large and negative. Since negative values of S have no physical significance, the critical values of S are presented only for positive values of S. The interpretation is that for R greater than a certain value (about 2000 in the present case) convection occurs no matter how small the value of S is. As R takes increasing negative values the stage is reached where the oscillatory mode comes into existence, and as R becomes more negative the frequency of oscillation increases monotonically from a zero value. Then, a value of R is reached at which the critical value for the oscillatory mode becomes less than the critical value for the monotonic mode. This means that as S increases the onset of convection will then appear as oscillatory motion. At the cross-over point in Fig. 2(a) there will be a jump in the observed wavenumber (see Fig. 2(b)) and the oscillatory convection will appear at a nonzero frequency. Thus the situation for biothermal convection is dramatically different from that for double diffusive convection (see, for example, [14, Section 9.1.1]). In the latter situation the oscillatory instability boundary bifurcates smoothly from the monotonic stability boundary and there is no jump in either wavenumber or frequency. It is obvious from Fig. 2(b) that the critical wavenumber is less (and so convection cells are wider) for the oscillatory mode in comparison with the monotonic mode at the same negative value of R, though still larger (narrower cells) in comparison with the value 3 (cell width approximately equal to layer thickness) that pertains when S has a small value. In interpreting Fig. 2(c) one should bear in mind that the frequency has been scaled in terms of the inverse of a time scale $\rho H^2/\mu$, that is in terms of a viscous diffusive time scale.

To see the effect of varying the gyrotactic number G we can compare Fig. 2 (for $G = 1$) with Fig. 3 (for $G = 10$) and Fig. 4 (for $G = 0.1$). The gyrotaxis number, G, characterizes the deviation of the cell swimming direction from strictly vertical.

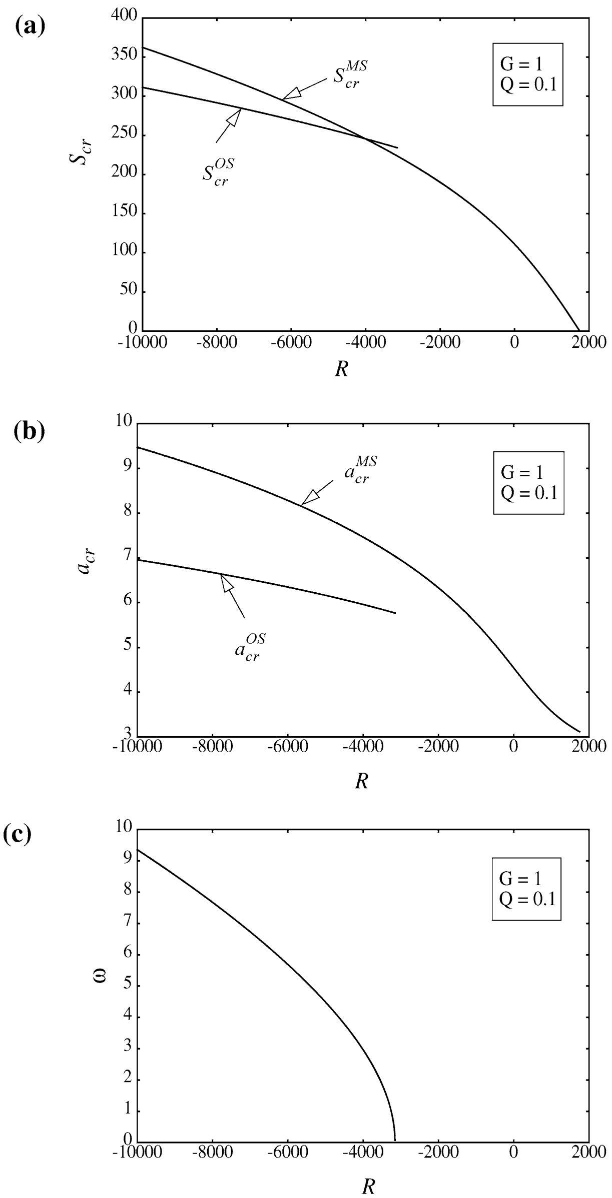

Fig. 5. As for Fig. 2, but with $Q = 0.1$ instead of $Q = 1$.

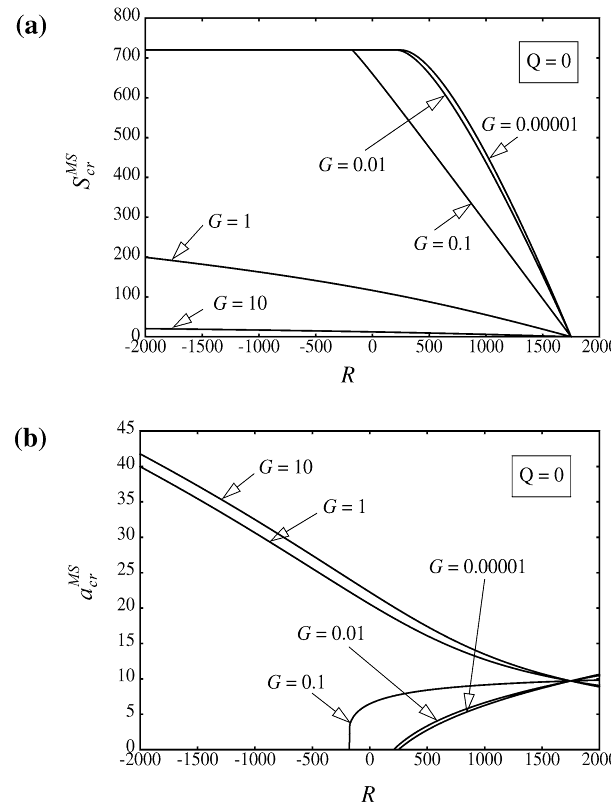

Fig. 6. Plots of (a) critical bioconvection Rayleigh number, (b) critical wavenumber, for monotonic stability, for the case $Q = 0$ and for various values of G.

If $G = 0$, there is no gyrotaxis and the microorganisms swim vertically upwards (exhibit negative geotaxis). Childress et al. [15] established that an infinite uniform suspension of negatively geotactic microorganisms ($G = 0$) is stable in the absence of cell concentration stratification. Pedley et al. [1] have shown that under the same conditions a suspension of gyrotactic microorganisms ($G > 0$) is unstable. Hence, gyrotaxis helps the development of convection instability.

The most dramatic change in Figs. 3 and 4 is in the scale on the S-axis. The critical bioconvection Rayleigh number decreases rapidly as G increases, approximately as $1/G$. Otherwise, there is very little change in going from $G = 1$ to $G = 10$, and only slightly more change in going from $G = 1$ to $G = 0.1$ (except for the appearance of an inflexion point in the curve for the critical wavenumber for the monotonic stability mode).

In considering the effect of varying the Péclet number Q, we found ourselves limited by the failure of our computer code to run for values of Q significantly greater than unity. This was not unexpected, because the eigenvalue equation involves exponential/hyperbolic functions of Q, and for large values of Q the basic number density profile is highly nonlinear, and so the whole basis of our approximate analysis could be expected to break down when Q becomes large. However, we can observe the effect of a reduction in the value of Q from 1 to 0.1 by comparing Fig. 5 with Fig. 2. Again there is little change, except for a modest increase in the values of the critical bioconvection Rayleigh number. When we further reduced Q to the value 0.01 we found numerical inaccuracy due to round-off error in computing the quotient of two small quantities. Since we had an asymptotic solution available for Q tending to zero, we did not persist with the difficult calculations.

For the limiting case $Q = 0$, Eqs. (56) and (57) apply. For this case we found no oscillatory convection, and we hypothesize that none is possible, though we were unable to obtain a formal proof of this. Our results for the onset of monotonic convection are presented in Fig. 6. The rapid decrease of S_{cr} as G increases observed here is consistent with the trend observed previously for the case of $Q = 1$. There is a range of values of G including the value 0.1 for which there is a kink in the stability curve, and this is associated with a rapid increase in the critical wavenumber from zero to a nonzero value. When $G = 0$, S takes the value 720 and a takes the value zero, independent of R, for all sufficiently small positive values and all negative values of R, and this occurs when the critical wavenumber is zero. Similar behaviour was observed by one of the present authors, in connection with a double-diffusive instability problem, some forty years ago [13].

4. Conclusions

We have applied a linear stability analysis to the problem of the onset of bioconvection in a thermally stratified fluid. The stability boundary depends on the values of the Lewis and Prandtl numbers, but these parameters were given fixed typical values for the presentation here. For the case of nonzero Péclet number, with bottom heating of the layer, oscillatory convection can be the favored mode of instability. The change in favored mode from monotonic to oscillatory is accompanied by a jump to a smaller value of the wavenumber and a jump in frequency from zero to a finite nonzero value. The critical bioconvection Rayleigh number decreases rapidly as the gyrotactic number increases, and it decreases less rapidly as the Péclet number increases.

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